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1980 J. Phys. A: Math. Gen. 13 2703

(http://iopscience.iop.org/0305-4470/13/8/020)

J. Phys. A: Math. Gen. **13** (1980) 2703–2722. Printed in Great Britain

On the gravitational radiation formula

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Received 19 November 1979, in final form 19 January 1980

Abstract. For electromagnetically as well as gravitationally bound quantum mechanical many-body systems the coefficients of absorption and induced emission of gravitational radiation are calculated in the first-order approximation. The results are extended subsequently to systems with arbitrary non-Coulomb-like two-particle interaction potentials; it is shown explicitly that in all cases the perturbation of the binding potentials of the bound systems by the incident gravitational wave field itself must be taken into account.

With the help of the thermodynamic equilibrium of gravitational radiation and quantised matter, the coefficients for spontaneous emission of gravitational radiation are derived and the gravitational radiation formula for emission of gravitational quadrupole radiation by bound quantum mechanical many-body systems is given. According to the correspondence principle our result is completely identical with the well known classical radiation formula, by which recent criticism against this formula is refuted.

Finally the quantum mechanical absorption cross section for gravitational quadrupole radiation is deduced and compared with the corresponding classical expressions. As a special example the vibrating two-mass quadrupole is treated explicitly.

1. Introduction

With the discovery of the orbital phase shift of the binary pulsar PSR 1913+16 by Taylor *et al* (1979) the formula for emission of gravitational quadrupole radiation has obtained essential significance. The expression for the average rate of energy loss of a physical system due to gravitational quadrupole radiation has been given already by Einstein in 1918. His result, which has appeared in several textbooks (see e.g. Landau and Lifshitz 1975), has the form

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -A\frac{f}{c^5}\overline{\frac{\mathrm{d}^3Q_{ab}}{\mathrm{d}t^3}\frac{\mathrm{d}^3Q^{ab^t}}{\mathrm{d}t^3}}, \qquad A = \tfrac{1}{45}, \tag{1.1}$$

where Q_{ab} is the mass quadrupole tensor of the radiation source and f is the Newtonian gravitational constant. Some time ago, however, Ehlers *et al* (1976) argued that this formula has not yet been derived either exactly or by means of a consistent approximation method for integration of Einstein's field equations of gravitation. Their conclusion is that therefore the value of the factor A in (1.1) is very uncertain. As far as we know this situation has not been changed until now.

In this paper we point to the possibility of proving equation (1.1) in a way different from the usual ones which avoids the difficulty of integration of the inhomogeneous field equations of gravitation. Instead of constructing a *classical* bound system as radiation source and integrating Einstein's field equations, we start from a *quantum mechanical* bound system described by the Schrödinger limit of the Klein–Gordon theory. Then we disturb it by an incident plane gravitational wave satisfying Einstein's vacuum field equations, whereby we restrict ourselves to weak gravitational waves in the framework of the linearised theory. By the time-dependent perturbation method the transition probabilities of absorption and induced emission of gravitational radiation can be calculated. This has been done already by several authors using special systems, for instance a harmonic oscillator (Gräfe and Dehnen 1976) or the hydrogen atom with fixed nucleus (Röpke 1972, Vinet 1977). In contrast to this it is our aim to start from an *arbitrarily* (electromagnetically, gravitationally, etc) bound and closed quantum mechanical system and to deduce from its transition probabilities for absorption and induced emission the transition probabilities of *spontaneous* emission of gravitational radiation†. This is possible in a simple manner—following Einstein's considerations for the electromagnetic case—by the assumption of thermodynamic equilibrium between gravitational radiation and bound quantum mechanical matter systems. In doing so we restrict ourselves in the following to the quadrupole approximation level.

For the special case of a linear harmonic quantum mechanical oscillator Gräfe and Dehnen (1976) have performed just such a calculation. Unfortunately they have assumed a 'rigid' oscillator, the internal potential of which is not influenced by the external gravitational wave field. As we show in the following, such an assumption is physically unrealistic. A gravitational influence on the internal potentials is always present, in consequence of which the result of Gräfe and Dehnen is only correct qualitatively but not quantitatively; it is smaller by a factor four than the exact one. In the paper of Vinet (1977) the influence of the gravitational wave field on the Coulomb potential of the hydrogen atom is neglected also; a correct consideration of the perturbation of the Coulomb field by the gravitational wave field is given by Röpke (1972).

By our procedure, which seems to be more simple than the classical absorption and emission problem, the formulae for *absorption* as well as for *emission* of gravitational radiation can be derived simultaneously even for *classical* systems, if one goes over finally to the classical limit according to the correspondence principle. We find agreement with the classical result for emission and for absorption, and the usual phenomenological acceleration formula for the relative distance x^a of two oscillator masses (see e.g. Misner *et al* 1973)

$$\ddot{x}^a + \omega_0^2(x^a - 2L^a) = -{}^{TT}R^a{}_{4b4}x^b \tag{1.2}$$

(${}^{TT}R^\alpha{}_{\beta\gamma\delta}$ is the Riemann tensor of the *TT*-gauged gravitational wave field; $2L^a$ is the rest distance of the oscillator masses) can be established explicitly on the basis of a microscopic model.

In order to facilitate the comparison of the classical emission formula with the quantum mechanical result it is useful to expand the quadrupole tensor in (1.1) according to the Fourier theorem

$$Q_{ab} = \sum_j [Q_{ab}(\omega_j)\,e^{i\omega_j t} + Q^*_{ab}(\omega_j)\,e^{-i\omega_j t}], \tag{1.3}$$

assuming a periodic source for the gravitational radiation. Then (1.1) results in

$$\frac{dE}{dt} = -\frac{2}{45}\frac{f}{c^5}\sum_j \omega_j^6 Q^{ab}(\omega_j)Q^*_{ab}(\omega_j). \tag{1.4}$$

† The quantum mechanical treatment of gravitational radiation from non-bound systems is performed especially by Weinberg (1965) and DeWitt (1967). See also Barker *et al* (1969).

This formula will be exactly confirmed by the detailed quantum mechanical calculation, if the well known substitution of the matrix elements through the corresponding Fourier coefficients is performed.

Finally we remark that our procedure is justified also by the fact that quantum theory possesses priority over the classical one and that therefore all classical physics and its results are to be deduced from quantum physics in the classical limit.

2. The interaction operator for gravitational wave fields

2.1. The external gravitational field

We use for the metric of space–time the linear approximation with respect to the flat metric†:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad g^{\mu\nu} = \eta^{\mu\nu} + \gamma^{\mu\nu},$$
$$|h_{\mu\nu}|, |\gamma^{\mu\nu}| \ll 1, \qquad \eta_{\mu\nu}, \eta^{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & +1 & & \\ & & +1 & \\ 0 & & & -1 \end{pmatrix}. \tag{2.1a}$$

Raising and lowering of tensor indices will be performed in the following by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ respectively. Then

$$\gamma^{\mu\nu} = -h^{\mu\nu} \tag{2.1b}$$

is valid. Additionally we demand the de Donder condition given by‡

$$h_\mu{}^\alpha{}_{|\alpha} - \tfrac{1}{2}h_{|\mu} = 0 \qquad (h = h_\mu{}^\mu). \tag{2.2}$$

Then Einstein's vacuum field equations take the form

$$h_{\mu\nu|\alpha}{}^{|\alpha} = 0. \tag{2.3}$$

The solution of (2.3) for a plane wave propagating in the direction of the x^3 axis can be written with respect to (2.2):

$$h_{11}(v) = -h_{22}(v) \neq 0, \qquad h_{12}(v) \neq 0, \qquad v = t - x^3/c,$$
$$h_{\mu\nu} = 0 \qquad \text{otherwise}. \tag{2.4}$$

A corresponding representation of $h_{\mu\nu}$ is valid, if the propagation direction is determined by the x^1 or the x^2 axis. Accordingly we obtain for the wave field from (2.2) and (2.4) within our approximation (*TT*-gauge):

$$h_{4\mu} = 0, \qquad h = 0, \qquad h_m{}^a{}_{|a} = 0, \qquad g = \det(g_{\mu\nu}) = -1. \tag{2.5}$$

2.2. The electromagnetically bound matter field

We consider first an electromagnetically coupled quantum mechanical system. To describe it we start from the general covariant minimally coupled Klein–Gordon

† Greek indices run from 1 up to 4, Latin indices from 1 up to 3. The fourth coordinate is $x^4 = ct$.
‡ $|\nu$ signifies the ordinary and $\|\nu$ the covariant partial derivative with respect to x^ν.

equation†

$$\psi^{|\mu}{}_{\|\mu} - 2\mathrm{i}(e/\hbar c)A^{\mu}\psi_{|\mu} - (e/\hbar c)^2 A^{\mu}A^{\nu}g_{\mu\nu}\psi - (mc/\hbar)^2\psi = 0, \tag{2.6}$$

where the Lorentz condition

$$A^{\mu}{}_{\|\mu} = 0 \tag{2.7}$$

for the electromagnetic 4-potential A^{μ} is used. Going over to the Schrödinger equation we substitute

$$\psi(x^{\mu}) = \exp[-\mathrm{i}(mc^2/\hbar)t]\Phi(x^{\mu}) \tag{2.8}$$

and neglect all terms up to the explicit order of c^{-2}. Then we get from (2.6), in the case of the linearised plane gravitational wave, using (2.1) and (2.5):

$$\frac{\hbar^2}{2m}\Phi_{|a}{}^{|a} - eA^4\Phi - \mathrm{i}\frac{e\hbar}{mc}A^a\Phi_{|a} - \frac{\hbar^2}{2m}h^{ab}\Phi_{|a|b} = -\mathrm{i}\hbar\frac{\partial\Phi}{\partial t} \tag{2.9}$$

with the magnetic vector potential A^a and the electric potential A^4. One obtains the same result in the Schrödinger limit starting from the general covariant Dirac equation.

The last term on the left-hand side of equation (2.9) represents the explicit deviation from the Schrödinger equation in flat space–time in consequence of the interaction with the gravitational wave field. However the foregoing terms containing the electric potential A^4 and the magnetic potential A^a are also disturbed by the gravitational wave, because the 4-potential A^{μ} is influenced by the gravitational wave via the covariant Maxwell equations. This is the reason for the fact that all calculations with rigid internal potentials of the matter system are inconsistent.

Accordingly we set

$$A^a = \overset{(0)}{A}{}^a + \overset{(1)}{A}{}^a, \qquad A^4 = \overset{(0)}{A}{}^4 + \overset{(1)}{A}{}^4, \tag{2.10}$$

where $\overset{(1)}{A}{}^a$ and $\overset{(1)}{A}{}^4$ represent the perturbation of the 4-potential caused by the wave in first order of h_{ab}. Then equation (2.9) reads

$$\frac{\hbar^2}{2m}\Phi_{|a}{}^{|a} - e\overset{(0)}{A}{}^4\Phi - \mathrm{i}\frac{e\hbar}{mc}\overset{(0)}{A}{}^a\Phi_{|a} - G\Phi = -\mathrm{i}\hbar\frac{\partial\Phi}{\partial t} \tag{2.11}$$

with the first-order perturbation operator

$$G = \frac{\hbar^2}{2m}h^{ab}\,\partial_a\,\partial_b + e\overset{(1)}{A}{}^4 + \mathrm{i}\frac{e\hbar}{mc}\overset{(1)}{A}{}^a\,\partial_a. \tag{2.11a}$$

In view of the last two terms there exists an amplification of the gravitational influence on a quantum mechanical system by the perturbation of the (binding) electromagnetic fields (enhancement of the result of Gräfe and Dehnen (1976) by a factor four).

Neglecting all magnetic effects in view of the factor c^{-1}, we obtain finally in the lowest perturbation order

$$\frac{\hbar^2}{2m}\Phi_{|a}{}^{|a} - e\overset{(0)}{A}{}^4\Phi - \frac{\hbar^2}{2m}h^{ab}\Phi_{|a|b} - e\overset{(1)}{A}{}^4\Phi = -\mathrm{i}\hbar\frac{\partial\Phi}{\partial t}. \tag{2.12}$$

† The difference to conformal coupling plays no role in the following.

Accordingly the one-particle Hamiltonian has the form

$$H = H_0 + W \tag{2.12a}$$

with

$$H_0 = -(\hbar^2/2m)\,\partial_a\,\partial^a + \overset{(0)}{V}, \qquad \overset{(0)}{V} = e\overset{(0)}{A}{}^4 \tag{2.12b}$$

and the perturbation operator

$$W = (\hbar^2/2m) h^{ab}\,\partial_a\,\partial_b + \overset{(1)}{V}, \qquad \overset{(1)}{V} = e\overset{(1)}{A}{}^4. \tag{2.12c}$$

For calculation of the perturbation potentials $\overset{(1)}{A}{}^a$ and $\overset{(1)}{A}{}^4$ we start from the general covariant Maxwell equation

$$(-g)^{-1/2}(\sqrt{-g}F^{\mu\nu})_{|\nu} = 4\pi j^\mu \tag{2.13}$$

with

$$F_{\mu\nu} = A_{\nu|\mu} - A_{\mu|\nu}. \tag{2.13a}$$

With the use of the Lorentz condition (2.7) we obtain for the potentials (2.10), in the case of the wave metric (2.1) with the properties (2.5) (a dot means the partial derivative with respect to t),

$$\overset{(0)}{A}{}^\mu{}_{|\nu}{}^{|\nu} = -4\pi \overset{(0)}{j}{}^\mu, \tag{2.14a}$$

$$\overset{(1)}{A}{}^4{}_{|\nu}{}^{|\nu} = \overset{(0)}{A}{}^4{}_{|a|b} h^{ab} - \frac{1}{c}\overset{(0)}{A}{}_{a|b}\dot{h}_a{}^b - 4\pi\overset{(1)}{j}{}^4, \tag{2.14b}$$

$$\begin{aligned}\overset{(1)}{A}{}^a{}_{|\nu}{}^{|\nu} = -\frac{1}{c}\left(\overset{(0)}{A}{}^{4|b} - \frac{1}{c}\overset{(0)}{\dot{A}}{}^b\right)\dot{h}^a{}_b + \overset{(0)}{\dot{A}}{}^a{}_{|b|c} h^{bc} \\ + \overset{(0)}{A}{}^b{}_{|c}(h_b{}^{c|a} - h_b{}^{a|c} - h^{ac}{}_{|b}) - 4\pi\overset{(1)}{j}{}^a,\end{aligned} \tag{2.14c}$$

where the 4-current j^μ is decomposed into

$$j^\mu = \overset{(0)}{j}{}^\mu + \overset{(1)}{j}{}^\mu \tag{2.15}$$

analogously to the potentials.

Now we must restrict ourselves, in view of the following quantum mechanical approach, to point sources of the electromagnetic field localised in a finite region, i.e. to the special form of the 4-current (see Landau and Lifshitz 1975, p 256)

$$j^\mu(x^\nu) = \frac{1}{\sqrt{-g}}\sum_i q_i\,\delta(x^a - x^a_{(i)})\frac{\mathrm{d}x^\mu}{\mathrm{d}x^4} \tag{2.16}$$

with the electric point charges q_i at the positions $x^a_{(i)}$ †. From (2.16) it follows immediately up to the first-order approximation (see (2.5)) that

$$j^\mu(x^\nu) = \sum_i q_i\,\delta(x^a - x^a_{(i)})\frac{\mathrm{d}x^\mu}{\mathrm{d}x^4} = \overset{(0)}{j}{}^\mu(x^\nu) \tag{2.16a}$$

† The δ function is defined by $\int \delta(x^a - x'^a)\,\mathrm{d}^3x' = 1$.

and therefore in view of (2.15)

$$\overset{(1)}{j}{}^{\mu}(x^{\nu})=0. \tag{2.16b}$$

If we consider the sources of the electromagnetic field as classical ones, the solution of the classical equations of motion for the coordinates $x^{a}_{(i)}$ of the point charges would be necessary. But in contrast to this in the framework of a *closed* quantum mechanical system, which is our aim (see § 3), this is not correct because the motion of the charges is then determined by the many-particle Schrödinger equation in a self-consistent way.

By restricting ourselves to the non-magnetic limit (2.12) only the knowledge of A^4, i.e. of $\overset{(0)}{j}{}^4$ and $\overset{(1)}{j}{}^4$ (cf (2.14)), is necessary. Then the differential equations (2.14) are reduced to (2.14*a*) and (2.14*b*) diminished by the magnetic terms; in view of (2.16*a*) and (2.16*b*) we find

$$\overset{(0)}{A}{}^4{}_{|\nu}{}^{|\nu}=-4\pi\sum_i q_i\,\delta(x^a-x^a_{(i)}), \tag{2.17a}$$

$$\overset{(1)}{A}{}^4{}_{|\nu}{}^{|\nu}=\overset{(0)}{A}{}^4{}_{|a|b}h^{ab}. \tag{2.17b}$$

Evidently the assumption of a *rigid* potential (see e.g. Gräfe and Dehnen 1976, Vinet 1977) is incorrect; but the supposition of a rigid charge distribution (2.16*b*) is justified within the first approximation. The equations (2.17) have as solutions the well known retarded integrals; but in view of our limit (velocity of the point sources small compared with the velocity of light, wavelength of the gravitational radiation large compared with the domain of the distribution of the sources) we can neglect the retardation and get

$$\overset{(0)}{A}{}^4(x^{\nu})=\sum_i\frac{q_i}{|x^n-x^n_{(i)}|}, \tag{2.18a}$$

$$\overset{(1)}{A}{}^4(x^{\nu})=-\frac{1}{4\pi}h^{a'b'}(t)\int\frac{\overset{(0)}{A}{}^4{}_{|a'|b'}}{|x^n-x'^n|}\,\mathrm{d}^3x'. \tag{2.18b}$$

In order to make the integration problem (2.17*b*) in view of (2.18*a*) mathematically well defined it is necessary to avoid the singularity of $\overset{(0)}{A}{}^4$ at the positions of the sources and to damp the potential sufficiently in the space-like infinity. Accordingly we modify $\overset{(0)}{A}{}^4$ in the integral (2.18*b*) as follows:

$$\overset{(0)}{A}{}^4(x^{\nu})=\sum_i\frac{q_i\exp(-\mu|x^n-x^n_{(i)}|)}{(|x^n-x^n_{(i)}|^2+a^2)^{1/2}}, \tag{2.18c}$$

which goes over into the solution (2.18*a*) in the limit $\mu\to 0$, $a\to 0$. Then using the relation

$$\Delta'|x^a-x'^a|=2/|x^a-x'^a|$$

we obtain from (2.18*b*), with the help of Green's second formula,

$$\begin{aligned}\overset{(1)}{A}{}^4(x^{\nu})&=-\frac{1}{8\pi}h^{a'b'}(t)\int\overset{(0)}{A}{}^4{}_{|a'|b'}(x'^{\nu})\,\Delta'\,|x^n-x'^n|\,\mathrm{d}^3x'\\&=-\frac{1}{8\pi}h^{a'b'}(t)\int|x^n-x'^n|(\Delta'\overset{(0)}{A}{}^4(x'^{\nu}))_{|a'|b'}\,\mathrm{d}^3x'.\end{aligned} \tag{2.19}$$

The last integral is well defined also in the limit $\mu \to 0$, $a \to 0$. Herewith we can use (2.18a), and equation (2.19) yields

$$\overset{(1)}{A}_4(x^\nu) = \tfrac{1}{2} h^{a'b'}(t) \int |x^n - x'^n| \sum_i q_i \, \partial_{a'} \partial_{b'} \, \delta(x^n_{(i)} - x'^n) \, d^3x'. \tag{2.20}$$

By partial integration we find

$$\begin{aligned} \overset{(1)}{A}{}^4(x^\nu) &= \tfrac{1}{2} h^{ab}(t) \, \partial_a \, \partial_b \int |x^n - x'^n| \sum_i q_i \, \delta(x^n_{(i)} - x'^n) \, d^3x' \\ &= \tfrac{1}{2} h^{ab}(t) \, \partial_a \, \partial_b \sum_i q_i |x^n - x^n_{(i)}|. \end{aligned} \tag{2.21}$$

Performing the differentiations we obtain the final result:

$$\overset{(1)}{A}{}^4(x^\nu) = -\tfrac{1}{2} h_{ab}(t) \sum_i \frac{q_i (x^a - x^a_{(i)})(x^b - x^b_{(i)})}{|x^n - x^n_{(i)}|^3}. \tag{2.22}$$

The one-particle Hamiltonian (2.12b) and its perturbation (2.12c) are determined by (2.18a) and (2.22)†.

2.3. *The gravitationally bound matter field*

The gravitationally bound system can be treated similarly to the electromagnetically coupled quantum mechanical system. In this case the metric (2.1a) must be extended, in view of the Newtonian static first-order terms $\breve{u}_{\mu\nu}$ and their perturbations $v_{\mu\nu}$, by the gravitational wave field $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \breve{u}_{\mu\nu} + v_{\mu\nu}. \tag{2.23}$$

Here the Newtonian part of the metric has the following structure:

$$\begin{aligned} &\breve{u}_{11} = \breve{u}_{22} = \breve{u}_{33} = \breve{u}_{44} = -2U/c^2, \\ &\breve{u}_{\mu\nu} = 0 \qquad \text{for } \mu \neq \nu, \end{aligned} \tag{2.24}$$

where U is the scalar Newtonian potential determined by the differential equation

$$\Delta U = 4\pi f \sum_i m_i \, \delta(x^a - x^a_{(i)}) \tag{2.25a}$$

with the solution

$$U = -f \sum_i \frac{m_i}{|x^a - x^a_{(i)}|}. \tag{2.25b}$$

As in the electromagnetic case we have restricted ourselves to localised point masses m_i as sources for the Newtonian gravitational field. Evidently $\breve{u}_{\mu\nu}$ is of the order c^{-2}; furthermore it is de Donder gauged.

The wave field $h_{\mu\nu}$ is given as before by (2.2)–(2.5). Beyond this we need in the following the order of magnitude of $h_{\mu\nu}$ more precisely. Because the energy density u of the free radiation field is given in the sense of the second quantisation by the arbitrary

† The result (2.22) can be derived from (2.17b) also, neglecting the retardation, if one applies the $\partial_\mu \partial^\mu$-operator on (2.17$b$) and substitutes the right-hand side with regard to (2.17a). With the help of $\Delta\Delta |x^n - x'^n| = -8\pi \, \delta(x^n - x'^n)$ one finds immediately the solution (2.20).

number density n of gravitons multiplied by their energy $\hbar\omega$, the order of $h_{\mu\nu}$ can be considered from (4.12), in view of (4.1), as c^{-1} (the remaining quantities n, $\hbar$, ω and f have no influence on the power of c). Or on the more classical level: in the transition case the energy of the radiation field is proportional to the absolute value of the potential energy of the bound matter system. By averaging over an appropriate normalising volume we find for its density $\overline{(\nabla U)^2}/8\pi f$ (self-energy terms are neglected). Insertion into (4.12) yields the order of magnitude relation $A^2 \sim \overline{(\nabla U)^2}/\omega^2 c^2$. With $\omega r \sim v$, where in the sense of the correspondence principle r represents a mean diameter and v a typical intrinsic velocity of the system, it follows immediately that $A \sim [(\overline{(\nabla U)^2})^{1/2}/(v^2/r)]v/c$. Because the bracket has the order of the square root of the volume of the material system divided by the normalising volume (taking into account the comparability of gravitational force and centrifugal force), the order of magnitude of $h_{\mu\nu}$ is, in view of (4.1), given by v/c apart from the root of the ratio of the volumes in question.

For $v_{\mu\nu}$ we demand as for $h_{\mu\nu}$ and $\breve{u}_{\mu\nu}$ the de Donder condition:

$$v_{\mu}{}^{\alpha}{}_{|\alpha} - \tfrac{1}{2}v_{|\mu} = 0 \qquad (v = v^{\alpha}{}_{\alpha}). \tag{2.26}$$

The magnitude of the perturbation $v_{\mu\nu}$ will be found to be of the order of $\breve{u}_{\alpha\beta}h_{\gamma\delta}$, and thus of c^{-3}, under consideration of the differential equations for $v_{\mu\nu}$ following from Einstein's field equations, taking into account all terms up to the order c^{-3}, which implies at most linear in $h_{\mu\nu}$. This means in the sense of second quantisation restriction to *one*-graviton processes only.

With respect to the magnitudes of $h_{\mu\nu}$, $\breve{u}_{\mu\nu}$ and $v_{\mu\nu}$ the inverse metric up to the order of c^{-3} has the form

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} - (\breve{u}^{\mu\nu} - h^{\mu\alpha}h_{\alpha}{}^{\nu}) - (v^{\mu\nu} - h^{\mu\alpha}\breve{u}_{\alpha}{}^{\nu} - h^{\nu\alpha}\breve{u}_{\alpha}{}^{\mu} + h^{\mu\alpha}h_{\alpha}{}^{\beta}h_{\beta}{}^{\nu}). \tag{2.23a}$$

In the *same* approximation we go with the metric (2.23), (2.23a) into the Klein–Gordon equation (2.6), neglecting the electromagnetic vector potential A^{μ}. With the properties (2.2)–(2.5) and (2.23)–(2.26) and with the ansatz (2.8) we obtain the following Schrödinger equation up to the order of c^{-1}:

$$\frac{\hbar^2}{2m}\Phi_{|a}{}^{|a} + \frac{mc^2}{2}(\breve{u}_{44} + v_{44})\Phi - \frac{\hbar^2}{2m}h^{ab}\Phi_{|a|b} = -\mathrm{i}\hbar\frac{\partial\Phi}{\partial t}. \tag{2.27}$$

Accordingly the one-particle Hamiltonian has the form

$$H = H_0 + W \tag{2.28}$$

with

$$H_0 = -(\hbar^2/2m)\,\partial_a\,\partial^a + \overset{(0)}{V}, \qquad \overset{(0)}{V} = -\tfrac{1}{2}mc^2\breve{u}_{44} = mU \tag{2.28a}$$

and the perturbation operator

$$W = (\hbar^2/2m)h^{ab}\,\partial_a\,\partial_b + \overset{(1)}{V}, \qquad \overset{(1)}{V} = -\tfrac{1}{2}mc^2 v_{44}. \tag{2.28b}$$

The result (2.28) corresponds exactly to the one-particle Hamiltonian (2.12a) for the electromagnetic case.

To calculate the perturbation potential $\overset{(1)}{V}$ we start from Einstein's field equations

$$R_{\mu\nu} = (8\pi f/c^4)(T_{\mu\nu} - \tfrac{1}{2}Tg_{\mu\nu}) \tag{2.29}$$

where, with point masses as sources (cf Landau and Lifshitz 1975),

$$T_{\mu\nu}=\sum_i \frac{m_i c^2}{\sqrt{-g}} g_{\mu\rho} g_{\nu\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \frac{\mathrm{d}x^\sigma}{\mathrm{d}x^4} \delta(x^a - x^a_{(i)}) \qquad (\mathrm{d}\tau^2 = -\mathrm{d}s^2). \tag{2.29a}$$

In view of (2.28*b*), only the equation (2.29) with $\mu=\nu=4$ is relevant for the perturbation operator. In the order mentioned above (all terms up to the order c^{-3}) we obtain

$$R_{44}=(4\pi f/c^4)T_{44} \tag{2.30}$$

where, with respect to the properties of (2.23) and (2.29*a*),

$$R_{44}=-\tfrac{1}{2}\Delta\breve{u}_{44}-\tfrac{1}{2}v_{44|\alpha}{}^{|\alpha}+\tfrac{1}{2}h^{ab}\breve{u}_{44|a|b}, \tag{2.30a}$$

$$T_{44}=\sum_i m_i c^2\, \delta(x^a - x^a_{(i)}). \tag{2.30b}$$

With the use of (2.24) and (2.25*a*) we obtain from (2.30)–(2.30*b*) the following differential equation for v_{44}, which results from the homogeneous part of (2.30) only:

$$v_{44|\alpha}{}^{|\alpha}=h^{ab}\breve{u}_{44|a|b}. \tag{2.31}$$

This equation is identical with (2.17*b*), if one identifies v_{44} with $\overset{(1)}{A}{}^4$ and $\breve{u}_{44}$ with $\overset{(0)}{A}{}^4$. Because $\breve{u}_{44}$ is given according to (2.25*b*) (see (2.24)) by

$$\breve{u}_{44}=\frac{2f}{c^2}\sum_i \frac{m_i}{|x^a - x^a_{(i)}|} \tag{2.32}$$

we immediately obtain, in view of (2.18*a*), the solution v_{44} from equation (2.22), substituting $q_i \to (2f/c^2)m_i$:

$$v_{44}=-\frac{f}{c^2}h_{ab}(t)\sum_i \frac{m_i(x^a - x^a_{(i)})(x^b - x^b_{(i)})}{|x^n - x^n_{(i)}|^3}. \tag{2.33}$$

With (2.32) and (2.33) the one-particle Hamiltonian (2.28*a*) and its perturbation (2.28*b*) are given in complete analogy to the electromagnetic case. On the other hand, in (2.31) the nonlinearity of Einstein's field equations is used explicitly, confirming the fact that the gravitational radiation of gravitationally bound systems is already a nonlinear effect in the lowest-order approximation.

3. The many-particle Hamilton operator

3.1. The electrodynamic and gravitational case

In order to obtain a closed quantum mechanical system which interacts with the external gravitational wave field, we go over to the many-body problem. The corresponding many-particle Hamiltonians are, in view of (2.12*b*), (2.12*c*), (2.18*a*) and (2.22) and in view of (2.28*a*), (2.28*b*), (2.32) and (2.33), from the same structure in the electromagnetic and the gravitational case. One finds immediately by successive identification of the *one* particle with all other ones:

$$H_0=-\sum_i \frac{\hbar^2}{2m_i}\partial_{(i)a}\,\partial_{(i)}{}^a+\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}\frac{q_{(i)}q_{(j)}}{|x^n_{(i)}-x^n_{(j)}|}, \tag{3.1}$$

$$W = h_{ab}\Big[\sum_i \frac{\hbar^2}{2m_i} \partial^a_{(i)} \partial^b_{(i)} - \frac{1}{4} \sum_{\substack{i,j \\ i\neq j}} q_{(i)} q_{(j)} \frac{(x^a_{(i)} - x^a_{(j)})(x^b_{(i)} - x^b_{(j)})}{|x^n_{(i)} - x^n_{(j)}|^3}\Big], \tag{3.2}$$

where in the electromagnetic case $q_{(i)}$ are the charges q_i of the particles and in the gravitational case $q_{(i)}$ are connected with the masses m_i of the particles according to $q_{(i)} = \mathrm{i}\sqrt{f} m_i$.

Using the identity

$$\frac{m_i}{4\hbar^2}[H_0, [H_0, x^a_{(i)} x^b_{(i)}]]$$

$$= \frac{\hbar^2}{2m_i} \partial^a_{(i)} \partial^b_{(i)} - \frac{1}{4} \sum_{\substack{j \\ i\neq j}} q_{(i)} q_{(j)} \frac{x^b_{(i)}(x^a_{(i)} - x^a_{(j)}) + x^a_{(i)}(x^b_{(i)} - x^b_{(j)})}{|x^n_{(i)} - x^n_{(j)}|^3} \tag{3.3}$$

the perturbation operator (3.2) takes in view of $h^{ab} = h^{ba}$ the form

$$W = h_{ab} \sum_i \frac{m_i}{4\hbar^2}[H_0, [H_0, x^a_{(i)} x^b_{(i)}]]. \tag{3.4}$$

With the definition of the mass quadrupole tensor

$$Q^{ab} = \sum_i m_i (3x^a_{(i)} x^b_{(i)} - r^2_{(i)} \eta^{ab}) \tag{3.5}$$

we obtain from (3.4)

$$W = (h_{ab}/12\hbar^2)[H_0, [H_0, Q^{ab}]], \tag{3.6}$$

where the property $h^a{}_a = 0$ (see (2.5)) is used. Evidently the gravitational radiation field described by the symmetric rank-two tensor h_{ab} couples automatically to the mass quadrupole tensor Q_{ab} of the matter in the lowest approximation.

3.2. *Generalised two-particle interaction*

The foregoing result for the interaction operator (3.6) in the electromagnetic and gravitational case has an interesting feature, which allows a generalisation of formula (3.6) to other non-Coulomb-like two-particle interactions.

If we define the potential energy and its perturbation in (3.1) and (3.2) respectively by (cf (2.12*b*), (2.12*c*) and (2.28*a*), (2.28*b*))

$$\overset{(0)}{V} = \frac{1}{2} \sum_{\substack{i,j \\ i\neq j}} \frac{q_{(i)} q_{(j)}}{|x^n_{(i)} - x^n_{(j)}|}, \tag{3.7}$$

$$\overset{(1)}{V} = -\tfrac{1}{4} h_{ab} \sum_{\substack{i,j \\ i\neq j}} q_{(i)} q_{(j)} \frac{(x^a_{(i)} - x^a_{(j)})(x^b_{(i)} - x^b_{(j)})}{|x^n_{(i)} - x^n_{(j)}|^3}, \tag{3.8}$$

we find the simple relation

$$\overset{(1)}{V} = \sum_i \xi^a_{(i)} \partial_{(i)a} \overset{(0)}{V}, \qquad \xi^a_{(i)} = \tfrac{1}{2} h^a{}_b x^b_{(i)}. \tag{3.9}$$

Here the vector $\xi^a_{(i)}$ has the following meaning. In consequence of the gravitational wave the particle m_i at the position $x^a_{(i)}$ feels an acceleration with respect to the origin of the coordinate system; in the lowest order one finds (see e.g. Misner *et al* 1973)

$$\ddot{x}^a_{(i)} = -{}^{TT}R^a{}_{4b4}x^b_{(i)} = \tfrac{1}{2}\ddot{h}^a{}_b x^b_{(i)}. \tag{3.10}$$

In the first iteration step the integral of (3.10) has the form

$$\overset{(1)}{x}{}^a_{(i)} = \overset{(0)}{x}{}^a_{(i)} + \tfrac{1}{2}h^a{}_b \overset{(0)}{x}{}^b_{(i)} \tag{3.11}$$

with the initial condition $\overset{(1)}{x}{}^a_{(i)} = \overset{(0)}{x}{}^a_{(i)}$ for absent $h^a{}_b$. Accordingly the vector $\xi^a_{(i)}$ is the translation vector $\overset{(1)}{x}{}^a_{(i)} - \overset{(0)}{x}{}^a_{(i)}$ induced by the gravitational wave and (3.9) has the meaning of the Lie derivative of the potential energy $\overset{(0)}{V}$ in the direction of the translation vector $\xi^a_{(i)}$ †.

This geometrical interpretation of (3.9) seems to be so fundamental that it should not be restricted to the Coulomb-like interaction potentials. On the contrary, it should be valid for arbitrary quasi-static two-particle interaction forces. Then the generalised perturbation operator (3.2) reads

$$W = h_{ab} \sum_i \frac{\hbar^2}{2m_i} \partial^a_{(i)} \partial^b_{(i)} + \tfrac{1}{2}h^a{}_b \sum_i x^b_{(i)} \partial_{(i)a} \overset{(0)}{V} \tag{3.12}$$

with the two-particle interaction potential

$$\overset{(0)}{V} = \frac{1}{2} \sum_{\substack{i,j \\ i\neq j}} V_{ij}(|x^n_{(i)} - x^n_{(j)}|) \qquad (V_{ij} = V_{ji}). \tag{3.12a}$$

With the undisturbed Hamiltonian

$$H_0 = -\sum_i \frac{\hbar^2}{2m_i} \partial_{(i)a} \partial_{(i)}{}^a + \overset{(0)}{V} \tag{3.13}$$

we find analogously to (3.3) the identity

$$\frac{m_i}{4\hbar^2}[H_0, [H_0, x^a_{(i)}x^b_{(i)}]] = \frac{\hbar^2}{2m_i} \partial^a_{(i)} \partial^b_{(i)} + \frac{1}{4} \sum_{\substack{j \\ i\neq j}} (x^b_{(i)} \partial^a_{(i)} V_{ij} + x^a_{(i)} \partial^b_{(i)} V_{ij}). \tag{3.14}$$

Inserting (3.14) into (3.12), we obtain with the definition (3.5) the result

$$W = (h_{ab}/12\hbar^2)[H_0, [H_0, Q^{ab}]] \tag{3.15}$$

in full accordance with (3.6), but for arbitrary two-particle interaction potentials V_{ij} (cf (3.12a) and (3.13)).

4. Transition probabilities and absorption and emission power

For the incident gravitational radiation we choose a superposition of compactly neighbouring monochromatic waves. In view of the fact that in the lowest approximation (quadrupole approximation) the space dependence of the wave can be neglected

† Note that in the non-relativistic limit the potential has the transformation property of a scalar quantity.

over the size of the quantum mechanical system, we set

$$h^{ab}(t)=\frac{1}{\sqrt{2}}\sum_j A_{(j)}[e^{ab}_{(j)(\alpha)}\exp(-\mathrm{i}\omega_j t-\mathrm{i}\delta_j)+e^{*ab}_{(j)(\alpha)}\exp(\mathrm{i}\omega_j t+\mathrm{i}\delta_j)] \tag{4.1}$$

with (see (2.5))

$$e^{a}_{(j)(\alpha)a}=0, \qquad e^{ab}_{(j)(\alpha)}k_{(j)b}=0, \qquad e^{ab}_{(j)(\alpha)}e^{*}_{(j)(\beta)ab}=\delta_{\alpha\beta}. \tag{4.1a}$$

Here the vector $k_{(j)}{}^{a}$ is the wavevector and $e^{ab}_{(j)(\alpha)}$ represents the polarisation tensor of the wave with frequency ω_j and arbitrary phase δ_j; the index (α) signifies the two possible polarisation states and $A_{(j)}$ means the real amplitudes of the monochromatic waves.

Furthermore we assume that the energy eigenvalue problem of the undisturbed quantum mechanical system

$$H_0\Phi_A=E_A\Phi_A, \qquad \Phi_A=\phi_A(x^a)\exp(-\mathrm{i}E_A t/\hbar) \tag{4.2}$$

is solved with the Hamiltonian H_0 according to (3.13) (A is the collecting index for all quantum numbers; Φ_A is a complete orthonormal function system). Then the full Schrödinger equation

$$(H_0+W)\Phi=\mathrm{i}\hbar\,\partial\Phi/\partial t \tag{4.3}$$

with the perturbation operator (3.15) will be solved with the help of the time-dependent perturbation ansatz

$$\Phi=\sum_A a_A(t)\Phi_A. \tag{4.3a}$$

Using (4.1) and (4.2) one obtains, by a first iteration step for the transition probability (per time unit) for absorption and induced emission of gravitational radiation with the polarisation tensor $e^{ab}_{(j)(\alpha)}$,

$$\begin{aligned} W_{A\rightleftarrows B}(e^{ab}_{(\alpha)}) &= \lim_{t\to\infty}\frac{1}{t}|a_{AB}|^2 \\ &= \lim_{t\to\infty}\sum_j \frac{4}{\hbar^2}|\langle\phi_B|\tilde{W}_j|\phi_A\rangle|^2 A^2_{(j)}\frac{\sin^2\frac{1}{2}(\omega_{AB}-\omega_j)t}{(\omega_{AB}-\omega_j)^2 t} \\ &= (2\pi/\hbar^2)\sum_j|\langle\phi_B|\tilde{W}_j|\phi_A\rangle|^2 A^2_{(j)}\,\delta(\omega_{AB}-\omega_j) \end{aligned} \tag{4.4}$$

with

$$\tilde{W}_j=\frac{e^{ab}_{(j)(\alpha)}}{12\sqrt{2}\hbar^2}[H_0,[H_0,Q_{ab}]] \tag{4.4a}$$

and the resonance frequency

$$\omega_{AB}=(E_B-E_A)/\hbar. \tag{4.4b}$$

For simplicity we have assumed that the initial and final states, ϕ_A and ϕ_B respectively, are non-degenerate. Insertion of (4.4a) into (4.4) gives, in view of (4.4b) and with respect to the fact that ϕ_A and ϕ_B are energy eigenstates of the Hamiltonian H_0 (see (4.2))

$$W_{A\rightleftarrows B}(e^{ab}_{(\alpha)})=\frac{\pi}{144\hbar^2}\omega^4_{AB}\sum_j|e^{ab}_{(j)(\alpha)}\langle\phi_B|Q_{ab}|\phi_A\rangle|^2 A^2_{(j)}\,\delta(\omega_{AB}-\omega_j). \tag{4.5}$$

These transition probabilities are independent of the choice of the origin of the coordinate system in case of internal excitations of the matter systems.

Now we consider two extensions of the result (4.5).

(a) If we introduce an internal damping of the quantum mechanical system represented by the damping constant Γ (finite lifetimes of the states ϕ_A and ϕ_B), the δ function in (4.5) must be substituted as follows:

$$\delta(\omega_{AB}-\omega_j)\rightarrow\frac{1}{\pi}\frac{\Gamma/2}{(\omega_{AB}-\omega_j)^2+\Gamma^2/4}. \tag{4.6}$$

(b) In order to become free from the special polarisation state $e^{ab}_{(j)(\alpha)}$ and the special direction of the wave propagation, we have to add up in (4.5) all polarisation states and to integrate over all directions. Consequently the following substitution is necessary:

$$|e^{ab}_{(j)(\alpha)}\langle\phi_B|Q_{ab}|\phi_A\rangle|^2\rightarrow Z\equiv\int\sum_\alpha e^{ab}_{(j)(\alpha)}e^{*cd}_{(j)(\alpha)}\langle\phi_B|Q_{ab}|\phi_A\rangle\langle\phi_B|Q_{cd}|\phi_A\rangle^*\,\mathrm{d}\Omega \tag{4.7}$$

with the solid angle element $\mathrm{d}\Omega$. Besides the properties (4.1a) the polarisation tensor fulfils the following completeness relation:

$$\begin{aligned}\sum_\alpha e_{(j)(\alpha)ab}e^*_{(j)(\alpha)cd}&=\tfrac{1}{2}(\hat{k}_a\hat{k}_b\hat{k}_c\hat{k}_d+\hat{k}_a\hat{k}_b\eta_{cd}+\hat{k}_c\hat{k}_d\eta_{ab}-\hat{k}_b\hat{k}_d\eta_{ac}-\hat{k}_b\hat{k}_c\eta_{ad}\\&\quad-\hat{k}_a\hat{k}_d\eta_{bc}-\hat{k}_a\hat{k}_c\eta_{bd}+\eta_{ac}\eta_{bd}+\eta_{ad}\eta_{bc}-\eta_{ab}\eta_{cd}),\end{aligned} \tag{4.8}$$

where $\hat{k}^a=k^a_{(j)}/\omega_j$ means the unit vector in the direction of the wavevector. Insertion of (4.8) into (4.7) results with respect to $Q_{ab}=Q_{ba}$ and $Q^a{}_a=0$ (cf (3.5)) in

$$Z=\tfrac{1}{2}\int(\hat{k}_a\hat{k}_b\hat{k}_c\hat{k}_d-4\hat{k}_a\hat{k}_c\eta_{bd}+2\eta_{ac}\eta_{bd})\langle\phi_B|Q^{ab}|\phi_A\rangle\langle\phi_B|Q^{cd}|\phi_A\rangle^*\,\mathrm{d}\Omega. \tag{4.9}$$

With the relations (see e.g. Weinberg 1972 § 10.5)

$$\int\hat{k}_a\hat{k}_b\hat{k}_c\hat{k}_d\,\mathrm{d}\Omega=\tfrac{4}{15}\pi(\eta_{ab}\eta_{cd}+\eta_{ac}\eta_{bd}+\eta_{ad}\eta_{bc}), \tag{4.10a}$$

$$\int\hat{k}_a\hat{k}_b\,\mathrm{d}\Omega=\tfrac{4}{3}\pi\eta_{ab} \tag{4.10b}$$

we find from (4.9) the result

$$Z=\tfrac{8}{5}\pi\langle\phi_B|Q^{ab}|\phi_A\rangle\langle\phi_B|Q_{ab}|\phi_A\rangle^*. \tag{4.11}$$

Finally we need in the following the energy density of the incident gravitational radiation (4.1). Starting from the Landau–Lifshitz energy pseudotensor we find with the use of (4.1a), after integration over all directions and adding up all polarisation states,

$$u=8\pi\sum_j(\omega_j^2c^2/32\pi f)A^2_{(j)}. \tag{4.12}$$

4.1. Transition probabilities for absorption and induced emission

In the limit of *vanishing damping* ($\Gamma=0$, see (4.6)) the following substitution must be performed:

$$A^2_{(j)}\rightarrow A^2(\omega)\,\mathrm{d}\omega,\qquad\sum_j\rightarrow\int_\omega, \tag{4.13}$$

whereby the expression (4.5) becomes well defined. In this way we obtain from (4.5)

$$W_{A\rightleftarrows B}(e^{ab}_{(\alpha)}) = (\pi/144\hbar^2)\omega^4_{AB}A^2(\omega_{AB})|e^{ab}_{(\alpha)}\langle\phi_B|Q_{ab}|\phi_A\rangle|^2 \tag{4.14}$$

as transition probability for the polarisation state $e^{ab}_{(\alpha)}$. In order to become free from the polarisation states and the propagation direction we have to make the substitution (4.7). With the result (4.11) we obtain the total transition probability

$$W_{A\rightleftarrows B} = (\pi^2/90\hbar^2)\omega^4_{AB}A^2(\omega_{AB})\langle\phi_B|Q^{ab}|\phi_A\rangle\langle\phi_B|Q_{ab}|\phi_A\rangle^*. \tag{4.15}$$

Applying the transition (4.13) to the energy density (4.12), we find the *spectral* energy density of the gravitational radiation ($u = \int \rho(\omega)\, d\omega$):

$$\rho(\omega) = 8\pi(\omega^2c^2/32\pi f)A^2(\omega). \tag{4.16}$$

According to this the quantity $A^2(\omega)$ in (4.14) and (4.15) can be eliminated. We find

$$W_{A\rightleftarrows B}(e^{ab}_{(\alpha)}) = \frac{\pi f}{36\hbar^2c^2}\omega^2_{AB}\rho(\omega_{AB})|e^{ab}_{(\alpha)}\langle\phi_B|Q_{ab}|\phi_A\rangle|^2, \tag{4.17a}$$

$$W_{A\rightleftarrows B} = \frac{2\pi^2 f}{45\hbar^2c^2}\omega^2_{AB}\rho(\omega_{AB})\langle\phi_B|Q^{ab}|\phi_A\rangle\langle\phi_B|Q_{ab}|\phi_A\rangle^*. \tag{4.17b}$$

The total absorption power of the quantum mechanical system is obtained by multiplication of (4.17*b*) by $\hbar\omega_{AB}$:

$$L_{A\rightarrow B} = \frac{2\pi^2 f}{45\hbar c^2}\omega^3_{AB}\rho(\omega_{AB})\langle\phi_B|Q^{ab}|\phi_A\rangle\langle\phi_B|Q_{ab}|\phi_A\rangle^*. \tag{4.18}$$

We note here that the result (4.18) can be compared with experiments in general only for the absorption from the ground state. Otherwise the induced emission of radiation in consequence of a transition from the state $|A\rangle$ into a lower energy state must be subtracted in order to obtain the observable absorption power. Only in the last case is the transition to the classical limit possible in general (cf van Vleck 1924).

By division of (4.17*a*) and (4.17*b*) by the spectral density (4.16) we obtain Einstein's transition probabilities for absorption and induced emission of gravitational radiation ($W = \rho B$):

$$B_{A\rightleftarrows B}(e^{ab}_{(\alpha)}) = \frac{\pi f}{36\hbar^2c^2}\omega^2_{AB}|e^{ab}_{(\alpha)}\langle\phi_B|Q_{ab}|\phi_A\rangle|^2 \tag{4.19a}$$

and

$$B_{A\rightleftarrows B} = \frac{2\pi^2 f}{45\hbar^2c^2}\omega^2_{AB}\langle\phi_B|Q^{ab}|\phi_A\rangle\langle\phi_B|Q_{ab}|\phi_A\rangle^*. \tag{4.19b}$$

This result is larger by a factor four than the corresponding one in the paper of Gräfe and Dehnen (1976).

4.2. *Transition probabilities for spontaneous emission*

With the help of thermodynamic equilibrium between the gravitational radiation and the quantum mechanical system, the relation between the already known coefficients $B_{A\rightleftarrows B}$ and the transition probabilities for spontaneous emission $A_{A\leftarrow B}$ can be derived easily. Because this is a crucial point of our considerations the single steps will be summarised briefly.

Considering two non-degenerate energy eigenstates E_A and E_B ($E_B > E_A$) of a quantised matter system, the equilibrium condition between matter and gravitational radiation reads with respect to spontaneous and induced emission and to absorption

$$(A_{A\leftarrow B} + \rho_{\text{th}} B_{A\leftarrow B}) \exp(-E_B/kT) = \rho_{\text{th}} B_{A\rightarrow B} \exp(-E_A/kT),$$

where ρ_{th} means the thermic spectral energy density of the radiation field and the exponential functions represent the occupation numbers of the energy states in question. The solution for $A_{A\leftarrow B}$ reads, with regard to (4.4b) and in view of $B_{A\leftarrow B} = B_{A\rightarrow B} = B_{A\leftrightarrows B}$,

$$A_{A\leftarrow B} = \rho_{\text{th}} B_{A\leftrightarrows B}[\exp(\hbar\omega_{AB}/kT) - 1]. \tag{4.20a}$$

To calculate $\rho_{\text{th}}(\omega_{AB})$ we start from the state density of the free gravitational radiation field. With respect to the stationary eigensolutions (for well defined boundary conditions) of the differential equation (2.3) and because of the two independent polarisation states this is given by ω_{AB}^2/π^2c^3. Furthermore we restrict ourselves to the classical Boltzmann limit $\hbar\omega_{AB}/kT \ll 1$. In this case every state of the radiation field has the mean energy kT, so that we obtain (Rayleigh–Jeans law)

$$\rho_{\text{th}} = (\omega_{AB}^2/\pi^2c^3)kT.$$

Insertion into (4.20a) and expansion of the exponential function results finally in

$$A_{A\leftarrow B} = (\hbar\omega_{AB}^3/\pi^2c^3)B_{A\leftrightarrows B}, \tag{4.20b}$$

which gives together with the *strong* equation (4.20a) the Planck distribution ρ_{th} for the thermic gravitational radiation. Inversely with the Planck distribution (Bose gas of gravitons) the result (4.20b) follows immediately from (4.20a). Evidently equation (4.20b) is based on simple but fundamental physical principles which the gravitational radiation must fulfil also.

Certainly the crucial relation (4.20b) is not restricted to thermodynamic equilibrium; we have used this only for deriving this general equation†. Together with (4.19) the transition probabilities for spontaneous emission of gravitational radiation by the quantum mechanical system have the form

$$A_{A\leftarrow B}(e_{(\alpha)}^{ab}) = (f/36\pi\hbar c^5)\omega_{AB}^5|e_{(\alpha)}^{ab}\langle\phi_B|Q_{ab}|\phi_A\rangle|^2 \tag{4.21a}$$

and

$$A_{A\leftarrow B} = (2f/45\hbar c^5)\omega_{AB}^5\langle\phi_B|Q^{ab}|\phi_A\rangle\langle\phi_B|Q_{ab}|\phi_A\rangle^*. \tag{4.21b}$$

Here the result (4.21a) belongs to the special polarisation mode $e_{(\alpha)}^{ab}$, whereas equation (4.21b) means the *total* spontaneous emission coefficient. A special application of formula (4.21b) to the hydrogen atom can be found in Weinberg (1972 § 10.8).

After multiplication of (4.21b) by the energy $\hbar\omega_{AB}$ (energy of the emitted gravitons) the total energy loss of the quantum mechanical system by gravitational quadrupole radiation results in

$$\frac{\mathrm{d}E}{\mathrm{d}t}_{A\leftarrow B} = -\frac{2f}{45c^5}\omega_{AB}^6\langle\phi_B|Q^{ab}|\phi_A\rangle\langle\phi_B|Q_{ab}|\phi_A\rangle^*. \tag{4.22}$$

This formula is in exact agreement with Einstein's classical result (1.4) with $\omega_i = \omega_{AB}$ in

† Of course relation (4.20b) can be deduced also by a full quantum-field theoretical calculation (see e.g. Schiff 1968, Weinberg 1972 § 10.8).

the sense of the correspondence principle, according to which in the classical limit the matrix elements of the quadrupole operator in (4.22) must be substituted by the corresponding Fourier coefficients of the quadrupole tensor. Accordingly the controversial factor $A=\frac{1}{45}$ in the classical gravitational radiation formula (1.1) is verified by our quantum mechanical calculation.

5. Absorption cross section

With the transition probability for absorption we are able to calculate the absorption cross section of the quantum mechanical system for gravitational radiation. Starting from a damped system, we find with the use of (4.5) and (4.6) for the transition probability of *absorption* for polarised directional radiation

$$W_{A\to B}(e^{ab}_{(\alpha)})=\frac{\omega^4_{AB}}{144\hbar^2}\sum_j |e^{ab}_{(j)(\alpha)}\langle\phi_B|Q_{ab}|\phi_A\rangle|^2A^2_{(j)}\frac{\Gamma/2}{(\omega_{AB}-\omega_j)^2+\Gamma^2/4}. \tag{5.1}$$

Restricting ourselves to one incident frequency $\omega_j=\omega$ and multiplying by the energy $\hbar\omega$, we obtain the absorption power

$$L_{A\to B}(\omega, e^{ab}_{(\alpha)})=\frac{\omega^4_{AB}\omega}{144\hbar}|e^{ab}_{(\alpha)}\langle\phi_B|Q_{ab}|\phi_A\rangle|^2A^2\frac{\Gamma/2}{(\omega_{AB}-\omega)^2+\Gamma^2/4}. \tag{5.2}$$

Elimination of A^2 by the energy density for polarised radiation per solid angle with frequency ω (cf (4.12))

$$u(e^{ab}_{(\alpha)})=(\omega^2c^2/32\pi f)A^2 \tag{5.3}$$

results in

$$L_{A\to B}(\omega, e^{ab}_{(\alpha)})=\frac{2\pi f}{9\hbar c^2}\frac{\omega^4_{AB}}{\omega}|e^{ab}_{(\alpha)}\langle\phi_B|Q_{ab}|\phi_A\rangle|^2u(e^{ab}_{(\alpha)})\frac{\Gamma/2}{(\omega_{AB}-\omega)^2+\Gamma^2/4}. \tag{5.4}$$

Division by the energy flux density of the radiation $(cu(e^{ab}_{(\alpha)}))$ gives finally the absorption cross section belonging to the polarisation mode $e^{ab}_{(\alpha)}$:

$$\sigma_{A\to B}(\omega, e^{ab}_{(\alpha)})=\frac{2\pi f}{9\hbar c^3}\frac{\omega^4_{AB}}{\omega}|e^{ab}_{(\alpha)}\langle\phi_B|Q_{ab}|\phi_A\rangle|^2\frac{\Gamma/2}{(\omega_{AB}-\omega)^2+\Gamma^2/4}. \tag{5.5}$$

Going over to unpolarised isotropic radiation, we have to perform in (5.5) the transition (4.7) and to average over the solid angle and the two polarisation states. Thus with the result (4.11) we find

$$\sigma_{A\to B}(\omega)=\frac{2\pi f}{45\hbar c^3}\frac{\omega^4_{AB}}{\omega}\langle\phi_B|Q^{ab}|\phi_A\rangle\langle\phi_B|Q_{ab}|\phi_A\rangle^*\frac{\Gamma/2}{(\omega_{AB}-\omega)^2+\Gamma^2/4} \tag{5.6}$$

in accordance with (4.18). In the resonance case ($\omega=\omega_{AB}$) we obtain

$$\sigma_{A\to B}(\omega_{AB})=\frac{4\pi f}{45\hbar c^3}\frac{\omega^3_{AB}}{\Gamma}\langle\phi_B|Q^{ab}|\phi_A\rangle\langle\phi_B|Q_{ab}|\phi_A\rangle^*. \tag{5.6a}$$

We note that the results (5.5) and (5.6) can be compared as the result (4.18) with experiments in general only for absorption processes from the ground state, because otherwise the induced emission in consequence of a transition from the state $|A\rangle$ into a lower energy state must be subtracted. This is also the reason for the fact that the direct transition from (5.5) and (5.6) to the classical limit by substitution of the matrix elements through the Fourier coefficients of the mass quadrupole tensor is impossible.

Finally we point out that the ω-dependence of our result (5.6) is in view of the factor ω^{-1} not significant, as can be seen on the level of the Lagrangian formalism. In the sense of the possibility of adding a total time derivative to the Lagrangian the factor ω^{-1} is not gauge invariant, so that ω^{-1} should be taken as ω_{AB}^{-1} and the result (5.6) has meaning only near the resonance point.

6. Comparison with the classical approach

In § 4.2 we have shown as the main result that the classical radiation formula for the emission of gravitational quadrupole radiation can be deduced exactly by our quantum mechanical approach. On the other hand, for the absorption of gravitational radiation no *general* transitions to classical results for the absorption power as well as the absorption cross section exist. A special transition will be treated in the following by the use of a simple model for the quantum mechanical system.

First, however, we show the general connection between our perturbation operator (3.15) and the phenomenological classical tidal force of equation (1.2). In the Heisenberg picture the cperator (3.15) reads within our approximation

$$W=-\frac{h_{ab}}{12}\frac{\mathrm{d}^2}{\mathrm{d}t^2}Q^{ab} \tag{6.1}$$

and would enter the Lagrangian as $-W$. Using the freedom of adding a total time derivative (cf the end of § 5) the operator (6.1) is equivalent to the operator (we choose the same symbol)

$$W=-\tfrac{1}{12}\ddot{h}_{ab}Q^{ab}. \tag{6.2}$$

According to (3.10),

$$\ddot{h}_{ab}=-2\,{}^{TT}R_{a4b4} \tag{6.3}$$

is valid, by which equation (6.2) takes the gauge invariant form (against infinitesimal coordinate transformations)

$$W=\tfrac{1}{6}\,{}^{TT}R_{a4b4}Q^{ab}. \tag{6.4}$$

Because of the tracelessness of (6.3), we obtain with the use of (3.5)

$$W=\tfrac{1}{2}\,{}^{TT}R_{a4b4}\sum_i m_i x^a_{(i)}x^b_{(i)}. \tag{6.5}$$

The classical force on the mass m_i corresponding to the operator (6.5) is in view of the Hamilton equations given by

$$-\partial W/\partial x^a_{(i)}=-{}^{TT}R_{a4b4}m_i x^b_{(i)} \tag{6.6}$$

in full accordance with the tidal force of equation (1.2). As can be easily seen, the same force follows also immediately from the Lagrangian formalism with second time derivatives, using $-W$ according to (6.1) as perturbation term. From this point of view it is to be expected that in the classical limit our quantum mechanical results for the absorption process are in exact agreement with the classical ones based on formula (1.2).

In order to show this feature explicitly we take as special model a linear harmonic oscillator. For this we choose two point masses m_1 and m_2 on the x^1 axis at the positions $x^1_{(1)}$ and $x^1_{(2)}$ with the rest distance $2L$ and the harmonic spring constant κ†. The appertaining two-particle Hamilton operator reads, according to (3.13),

$$H_0 = -\frac{\hbar^2}{2m_1}\partial_{(1)a}\,\partial_{(1)}{}^a - \frac{\hbar^2}{2m_2}\partial_{(2)a}\,\partial_{(2)}{}^a + \frac{\kappa}{2}(x^1_{(1)} - x^1_{(2)} - 2L)^2. \tag{6.7}$$

Going over to relative and centre of mass coordinates ($M = m_1 + m_2$)

$$x^a = x^a_{(1)} - x^a_{(2)}, \tag{6.8a}$$

$$X^a = M^{-1}(m_1 x^a_{(1)} + m_2 x^a_{(2)}), \tag{6.8b}$$

we obtain from (6.7)

$$\begin{aligned} H_0 &= -\frac{\hbar^2}{2M}\frac{\partial}{\partial X^a}\frac{\partial}{\partial X^b}\eta^{ab} + h_0, \\ h_0 &= -\frac{\hbar^2}{2\mu}\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^b}\eta^{ab} + \frac{\omega_0^2\mu}{2}(x^1 - 2L)^2, \end{aligned} \tag{6.9}$$

with the abbreviations

$$\mu = m_1 m_2/M \qquad \text{(reduced mass)}, \tag{6.9a}$$

$$\omega_0 = (\kappa/\mu)^{1/2} \qquad \text{(eigenfrequency)}. \tag{6.9b}$$

For calculation of the absorption cross section (5.6) the knowledge of the quadrupole tensor of the system is necessary. From (3.5) it follows with use of (6.8*a*) and (6.8*b*) that

$$\begin{aligned} Q^{ab} &= M(3X^aX^b - R^2\eta^{ab}) + q^{ab}, \\ q^{ab} &= \mu(3x^ax^b - r^2\eta^{ab}) \end{aligned} \tag{6.10}$$

with

$$R^2 = X^aX^b\eta_{ab}, \qquad r^2 = x^ax^b\eta_{ab}. \tag{6.10a}$$

As it should be the total Hamilton operator $H = H_0 + W$ (cf (3.13) and (3.15)) separates with respect to the centre of mass coordinates (external degrees of freedom) and the relative coordinates (internal degrees of freedom) and we can restrict ourselves for our purpose without loss of generality to the internal degrees of freedom. Then the accompanying energy eigenvalue equation reads (see (6.9))

$$h_0\phi_n = E_n\phi_n \tag{6.11}$$

with the well known oscillator energy eigenvalues and eigenvalue solutions‡. Accordingly we find in view of (5.6) and (6.10)

$$\begin{aligned} \langle\phi_B|q^{ab}|\phi_A\rangle\langle\phi_B|q_{ab}|\phi_A\rangle^* &= \tfrac{3}{2}\langle\phi_{n+1}|q^{11}|\phi_n\rangle\langle\phi_{n+1}|q_{11}|\phi_n\rangle^* \\ &= 48\mu L^2(n+1)\hbar/\omega_0 \end{aligned} \tag{6.12a}$$

† This oscillator corresponds exactly to the 'vibrator' treated by Misner *et al* (1973 § 37.5). At the same time it is an example for a system with non-Coulomb-like two-particle interaction.

‡ For shortness the quantum numbers of the plane wave in the x^2 and x^3 directions are omitted.

for $A=n$ and $B=n+1$,

$$\langle\phi_B|q^{ab}|\phi_A\rangle\langle\phi_B|q_{ab}|\phi_A\rangle^* = 48\mu L^2 n\hbar/\omega_0 \tag{6.12b}$$

for $A=n-1$ and $B=n$, and zero for other values of A and B, valid when $2L$ is large in comparison with the domain, where the wavefunctions are essentially different from zero†. Evidently in this case the selection rule reads $\Delta n=\pm 1$ and $\omega_{AB}=\omega_0$ holds.

Herewith we obtain from (5.6) for the absorption case

$$\sigma_{n\to n+1}(\omega)=\frac{32}{15}\frac{\pi f}{c^3}\mu L^2\frac{\omega_0^3}{\omega}\frac{(n+1)\Gamma/2}{(\omega_0-\omega)^2+\Gamma^2/4} \tag{6.13a}$$

and for the case of induced emission, analogously,

$$\sigma_{n-1\leftarrow n}(\omega)=\frac{32}{15}\frac{\pi f}{c^3}\mu L^2\frac{\omega_0^3}{\omega}\frac{n\Gamma/2}{(\omega_0-\omega)^2+\Gamma^2/4}. \tag{6.13b}$$

The *effective* absorption cross section for the state $|n\rangle$ is, according to the consideration following equation (5.6), given by

$$\begin{aligned}\sigma_n(\omega)&=\sigma_{n\to n+1}(\omega)-\sigma_{n-1\leftarrow n}(\omega)\\&=\frac{32}{15}\frac{\pi f}{c^3}\mu L^2\frac{\omega_0^3}{\omega}\frac{\Gamma/2}{(\omega_0-\omega)^2+\Gamma^2/4}.\end{aligned} \tag{6.14}$$

This result does not depend on $\hbar$ and the quantum number n and can immediately be interpreted classically. Furthermore it is identical with the pure absorption cross section (6.13a) for the ground state $n=0$.

Choosing an oscillator with equal masses $m_1=m_2=m$, the result (6.14) leads to

$$\sigma(\omega)=\frac{8}{15}\frac{\pi f}{c^3}mL^2\frac{\omega_0^3}{\omega}\frac{\Gamma}{(\omega_0-\omega)^2+\Gamma^2/4}. \tag{6.15}$$

This effective absorption cross section is identical with the result given by Misner *et al* (1973, p 1024).

On the other hand, we are now able to write down also the energy loss of the oscillator by gravitational radiation in a simple way. Insertion of the result (6.12b) into equation (4.22) gives immediately as the energy loss by spontaneous emission from the state $|n\rangle$

$$\left.\frac{dE}{dt}\right|_{n-1\leftarrow n}=-\frac{32}{15}\frac{f}{c^5}\hbar\omega_0^5\mu L^2 n. \tag{6.16}$$

Finally we go over to the classical limit. Choosing for simplicity equal masses $m_1=m_2=m$, we find by equating the energy eigenvalue for $n\gg 1$ with the energy of the classical oscillator with the same frequency

$$n\hbar=ml^2\omega_0, \tag{6.17}$$

where l means the classical amplitude of the single masses. Substituting $n\hbar$ in (6.16)

† This is equivalent to the condition $n\ll 2L^2\omega_0\mu/\hbar$.

according to (6.17) we obtain for the emitted power of the 'vibrator' due to gravitational radiation

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{16}{15}\frac{f}{c^5}m^2L^2l^2\omega_0^6 \tag{6.18}$$

in full accordance with the classical calculations (see e.g. Ohanian 1976).

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